Announcements

1) Final Exam Tuesday
April 24th 6:30-9:30 PM

2) Final Review?

3) Last EC - survey!

Survey on CTools
under "Test Center"

Arclength

(Sections 8.1, 10.2, 10.4)

Parametric curve given by

$x(t)$, $y(t)$ for $t_0 \le t \le t_1$.

Find the length of the curve!

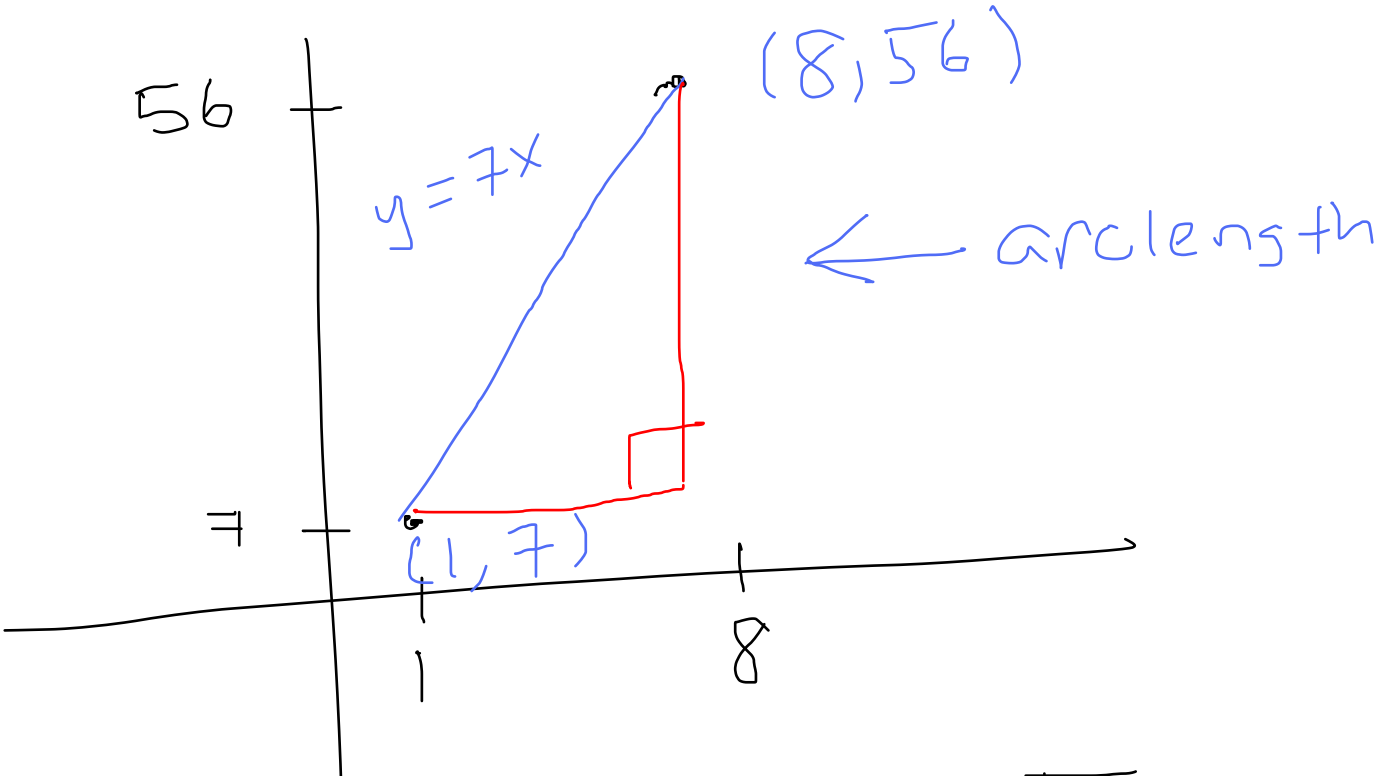
Example 1: Let $x(t) = t$,

$y(t) = 7t$.

Find the length of the curve from $t = 1$ to $t = 8$.

This is the line $y = 7x$ from $x = 1$ to $x = 8$.

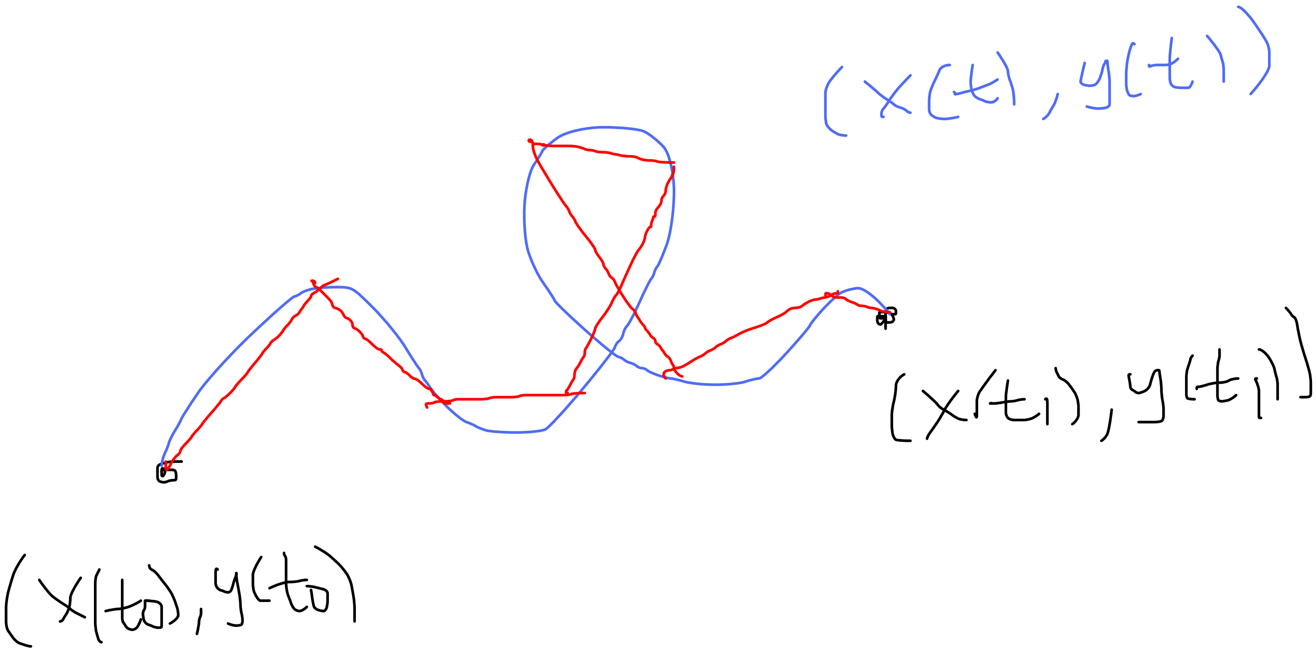
Picture



$$\text{arclength} = \sqrt{\underbrace{7}_{\text{x distance}}^2 + \underbrace{49}_{\text{y distance}}^2}$$

$$= 7\sqrt{1+49} = \boxed{7\sqrt{50}}$$

Picture (general)



Approximate the length using line segments, take limit as size of segments go to zero.

Formula:

$$L = \int_{t_0}^{t_1} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

provided $x'(t), y'(t)$ are continuous on $[t_0, t_1]$

Example 2: $x(t) = e^{3t} + e^{-3t}$

$y(t) = 24 - 6t$

from $t = 0$ to $t = \ln(8)$.

Use formula - don't forget to take derivatives!

$x'(t) = 3e^{3t} - 3e^{-3t}$ (chain rule)

$y'(t) = -6$

$$(x'(t))^2 + (y'(t))^2$$

$$= (3e^{3t} - 3e^{-3t})^2 + (-6)^2$$

$$= (9\underbrace{e^{3t}\cdot e^{3t}}_{= e^{6t}} - 18\underbrace{e^{3t}\cdot e^{-3t}}_{= e^{0}=1} + 9\underbrace{e^{-3t}\, e^{-3t}}_{= e^{-6t}}) + 36$$

$$= \underline{9e^{6t} - 18 + 9e^{-6t}} + 36$$

$$= \underline{9e^{6t} + 18 + 9e^{-6t}}$$

$$= (3e^{3t} + 3e^{-3t})^2$$

S_1

$$L = \int_0^{\ln(8)} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

$$= \int_0^{\ln(8)} \cancel{\sqrt{}}\,(3e^{3t} + 3e^{-3t})^{\cancel{2}}\, dt$$

$$= \int_0^{\ln(8)} (3e^{3t} + 3e^{-3t})\, dt$$

$$= \left(e^{3t} - e^{-3t}\right)\Big|_0^{\ln(8)}$$

$$= e^{3\ln(8)} - e^{-3\ln(8)}$$

$$= e^{\ln(8^3)} - e^{\ln(8^{-3})}$$

$$= 8^3 - \frac{1}{8^3}$$

$$= \boxed{512 - \frac{1}{512}}$$

Recall: Any function $y = f(x)$ may be parameterized

as $x(t) = t$, $y(t) = f(t)$.

Using the arclength formula,

$$x'(t) = 1, \quad y'(t) = f'(t)$$

Substitute the variable x for t to get

Cartesian Arclength

The arclength from $x=a$ to $x=b$ of $y=f(x)$ is

$$\boxed{L = \int_a^b \sqrt{1+(f'(x))^2}\,dx}$$

(need f' continuous on $[a,b]$)

Example 3: $y = x^2$

Arclength from $x=-1$ to $x=2$

$$f(x) = x^2$$

$$f'(x) = 2x \quad \text{(remember derivative)}$$

$$L = \int_{-1}^{2} \sqrt{1 + \boxed{(2x)^2}}\, dx$$

$$= \int_{-1}^{2} \sqrt{1 + 4x^2}\, dx$$

trig substitution! $2x = \tan(\theta)$

$$\int_{-1}^{2} \sqrt{1+4x^2}\, dx$$

$$2x = \tan\theta$$

$$x = \frac{\tan\theta}{2},$$

$$\boxed{dx = \frac{\sec^2\theta}{2}\, d\theta}$$

Temporarily ignore bounds, plug in.

$$\int \sqrt{1+(\tan\theta)^2} \cdot \frac{\sec^2\theta}{2}\, d\theta$$

$$= \int \sqrt{1+\tan^2\theta} \cdot \frac{\sec^2\theta}{2}\, d\theta$$

$$\int \sqrt{\underbrace{1+\tan^2\theta}_{1+\tan^2\theta=\sec^2\theta}} \cdot \frac{\sec^2\theta}{2}\, d\theta$$

$$= \frac{1}{2}\int \cancel{\sqrt{\sec^{\cancel{2}}\theta}} \cdot \sec^2\theta\, d\theta$$

$$= \frac{1}{2}\int \sec^3\theta\, d\theta$$

We already showed that

$$\int \sec^3\theta\, d\theta =$$

$$\frac{1}{2}\left(\ln|\sec\theta+\tan\theta| + \sec\theta\tan\theta\right)$$

So

$$\int \sqrt{1+4x^2}\,dx$$

$$= \frac{1}{2}\left(\frac{1}{2}\ln|\sec\theta + \tan\theta| + \sec\theta\tan\theta\right)$$

where $2x = \tan\theta$.

(substitution)

$$= \frac{1}{4}\ln|\sec\theta + 2x| + \sec\theta \cdot 2x$$

Since $\sec^2\theta = 1+\tan^2\theta$

$$= 1+(2x)^2$$

$$= 1+4x^2,$$

$$\sec\theta = \sqrt{1+4x^2}$$

Finally,

$$\int \sqrt{1+4x^2}\,dx$$

$$= \frac{1}{4}\left(\ln\left|\sqrt{1+4x^2}+2x\right| + 2x\sqrt{1+4x^2}\right),$$

so $\displaystyle\int_{-1}^{2} \sqrt{1+4x^2}\,dx$

$$\boxed{= \frac{1}{4}\left(\left(\ln(\sqrt{17}+4) + 4\sqrt{17}\right) - \left(\ln(\sqrt{5}-2) - 2\sqrt{5}\right)\right)}$$

I quit.

Polar Arclength

Remember $x = r\cos\theta$, $y = r\sin\theta$,

If $r = f(\theta)$, then

$x = f(\theta)\cos\theta$, $y = f(\theta)\sin\theta$

Change θ's to t's, apply parametric arclength formula.

Eventually, it simplifies to

$$\boxed{L = \int_{\theta_0}^{\theta_1} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta}$$

$(r = f(\theta))$

$$= \int_{\theta_0}^{\theta_1} \sqrt{(f(\theta))^2 + (f'(\theta))^2}\, d\theta$$

from $\theta = \theta_0$ to $\theta = \theta_1$

(provided f, f' continuous)

Example 4: $r(\theta) = 17\sin\theta$

$$r'(\theta) = 17\cos\theta$$

Arclength from $\theta = \frac{\pi}{2}$ to $\theta = 42$.

$$L = \int_{\frac{\pi}{2}}^{42} \sqrt{(r(\theta))^2 + (r'(\theta))^2}\, d\theta$$

$$= \int_{\frac{\pi}{2}}^{42} \sqrt{(17)^2 \sin^2\theta + (17)^2 \cos^2\theta}\, d\theta$$

$$= 17\int_{\frac{\pi}{2}}^{42} \sqrt{\underbrace{\sin^2\theta + \cos^2\theta}_{=1}}\, d\theta$$

$$= 17\int_{\frac{\pi}{2}}^{42} 1 \, d\theta$$

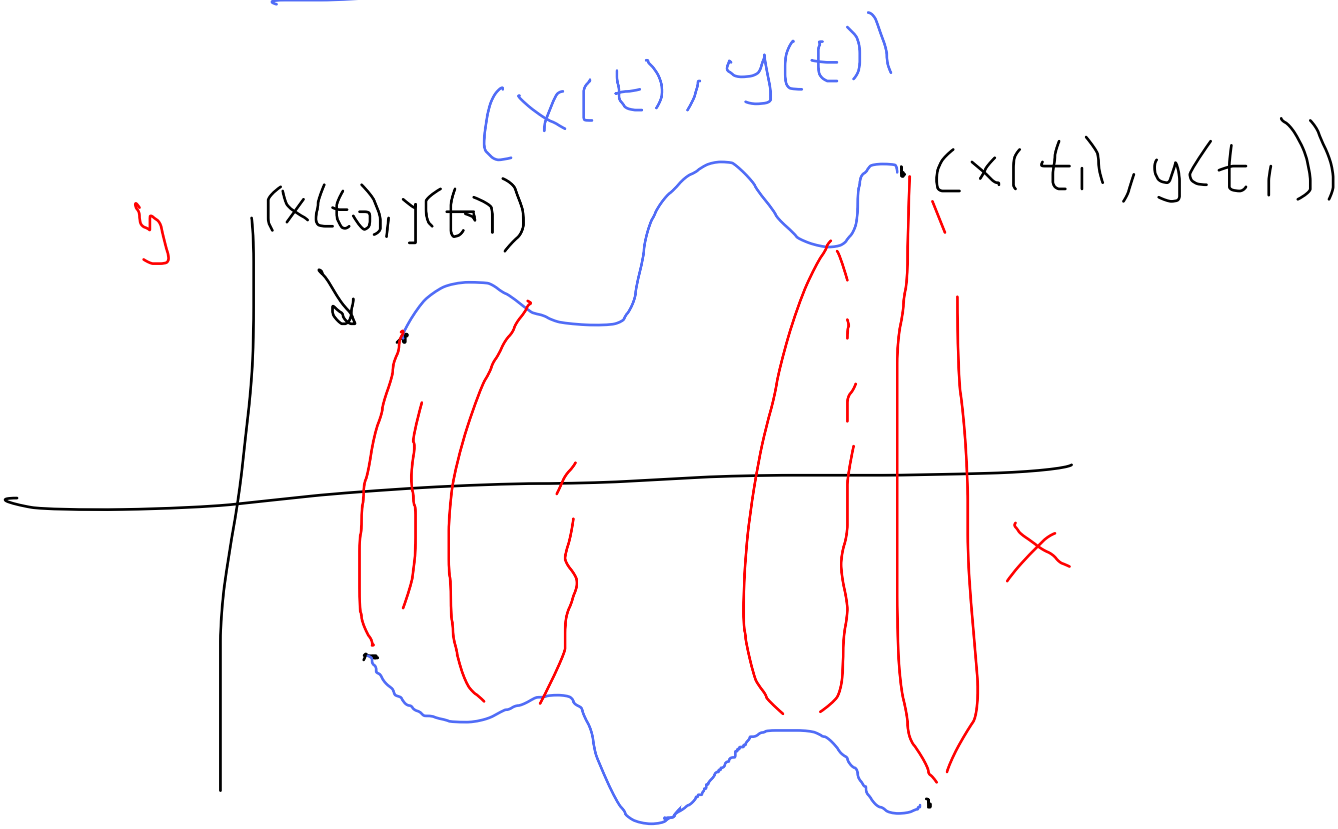
$$= \boxed{17\left(42 - \frac{\pi}{2}\right)}$$

Surface Area

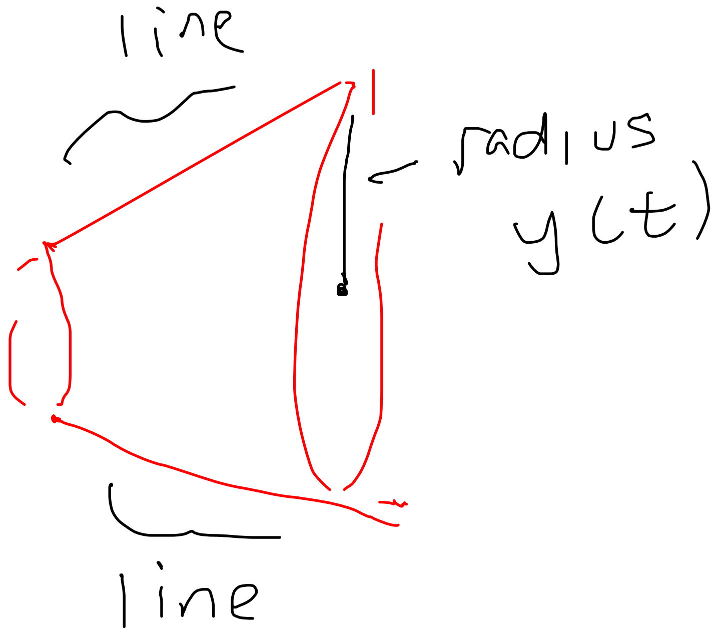
(Sections 10.2 + 8.2)

Idea: given a parametric curve $x(t)$, $y(t)$, revolve the portion from $t = t_0$ to $t = t_1$ about either x-axis or y-axis

Picture



Spin about x-axis.
approximate by line segments
like arclength. We get
a bunch of things like



Circumference of circle is

$2\pi\, y(t)$ – this shows up in formula

Parametric

About the x-axis from

$t = t_0$ to $t = t_1$

$$\boxed{SA = 2\pi \int_{t_0}^{t_1} y(t) \sqrt{(x'(t))^2 + (y'(t))^2}\, dt}$$

(x', y' continuous on $[t_0, t_1]$)

About the y-axis,

$$SA = 2\pi \int_{t_0}^{t_1} x(t) \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

Cartesian

Again using $x(t) = t$, $y(t) = f(t)$, we get the surface area from $x = a$ to $x = b$ obtained by spinning $y = f(x)$ about the

X-axis: $$SA = 2\pi \int_a^b f(x)\sqrt{1+(f'(x))^2}\, dx$$

y-axis: $$SA = 2\pi \int_a^b x\sqrt{1+(f'(x))^2}\, dx$$

Example 5: $f(x) = \sqrt{1+e^x}$

$0 \le x \le 1$

Surface area about x-axis!

$$SA = 2\pi \int_0^1 f(x)\sqrt{1+(f'(x))^2}\,dx$$

$$f(x) = (1+e^x)^{1/2}$$

$$f'(x) = \frac{1}{2}(1+e^x)^{-1/2} \cdot \underline{e^x}$$

chain rule

Plug in!

$$SA = 2\pi \int_0^1 \sqrt{1+e^x} \cdot \sqrt{1+\left(\frac{e^x}{2\sqrt{1+e^x}}\right)^2}\, dx$$

combine!

$$= 2\pi \int_0^1 \sqrt{1+e^x} \cdot \sqrt{1+\frac{e^{2x}}{4(1+e^x)}}\, dx$$

$$= 2\pi \int_0^1 \sqrt{(1+e^x) + \frac{\cancel{(1+e^x)} \cdot e^{2x}}{4\cancel{(1+e^x)}}}\, dx$$

$$= 2\pi \int_0^1 \sqrt{1+e^x+\frac{e^{2x}}{4}}\, dx$$

$$= 2\pi \int_0^1 \cancel{\sqrt{}}\left(1+\frac{e^x}{2}\right)^{\cancel{2}} dx$$

$$= 2\pi \int_0^1 \left(1 + \frac{e^x}{2}\right) dx$$

$$= 2\pi \left(x + \frac{e^x}{2}\right)\Big|_0^1$$

$$= 2\pi \left(\left(1 + \frac{e}{2}\right) - 1/2\right)$$

$$= 2\pi \left(\frac{1}{2} + \frac{e}{2}\right)$$

$$= \boxed{\pi + \pi e}$$